Modeling Error Bounds for Flexible Structures with Application to Robust Control

Yossi Chait*
University of Massachusetts, Amherst, Massachusetts 01003

Introduction

RECENT control theories for distributed parameter systems[1,2] can be used to synthesize robust controllers based on approximate models. Central to these theories is a standard error bound defined as the frequency response magnitude difference between the approximate and the actual open-loop models. For large space structures, this error bound has been derived to account for approximations of an infinite series of underdamped second-order vibration modes with an assumed uniform damping ratio. Unfortunately, this error bound has several limitations in large space structure applications, because it does not apply when the model has uncertainties in its coefficients. For example, it is well known from experiments that the uniform modal damping assumption is incorrect. A realistic analytical model should allow uncertainties in the modal damping ratio and in the location of the actuators and sensors.

Because of these uncertainties, a typical closed-loop control objective is to achieve uniform decay rate in all of the modes. Clearly, the standard error bound that is computed along the imaginary axis would not apply in such a situation. To achieve this control objective, the error bound must be computed along a specified vertical line in the open left half-plane.

In this Note, improved general formulas are developed for computing useful error bounds for approximations of certain models of large space structures. The new formulas are applicable to both theoretical (infinite-dimensional) models and models generated via identification or finite element techniques (which have a large, but finite, number of vibration modes). The bounds developed in this Note, in the spirit of earlier results,[3,4] can be used to specify the order of the initial (open-loop) approximate model and the minimum number of modes necessary for (closed-loop) control synthesis.

Frequency Domain Approximation Bounds

The approximation error bounds are developed for systems whose dynamics can be represented by a series solution, where each term in the series arises from a second-order ordinary differential equation. Consider the following frequency response function (FRF)

$$G(s) = \sum_{k=1}^{\infty} \frac{\delta_k}{s^2 + 2\zeta_k \omega_k s + \omega_k^2} \tag{1}$$

where $\{\delta_k\}$ are bounded real numbers $\{\omega_k = c_k k^\rho\}$ the natural frequencies (ρ and c_k are positive real), and $\{\zeta_k\}$ the modal damping factors for underdamped transfer functions with overshoot: $0 < \epsilon_1 \le \zeta_k \le \epsilon_2 < 0.707$, $k = 1,2,\ldots$. Note that $\{\delta_k\}$ are equal to the product of the eigenfunctions at the sensor and actuator locations, that if $\zeta_k > 0.707$ then a resonant peak does not exist (the bounds then can be computed as shown in Chait et al.[2]), and that the assumption of the c_k term in the natural frequencies can be modified easily without loss of generality.

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*Assistant Professor, Mechanical Engineering.

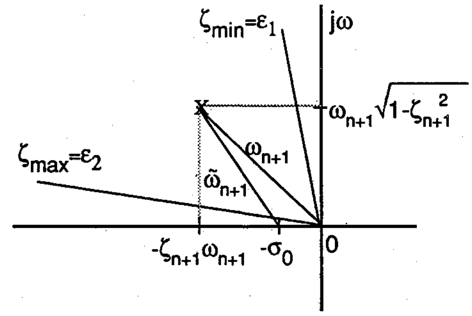

Fig. 1 Modified second-order root.

When the control objective is to achieve a certain response decay rate, say, σ_0 (i.e., closed-loop exponential stability), the transfer function [Eq. (1)] becomes

$$G(s - \sigma_0) = \sum_{k=1}^{\infty} \frac{\delta_k}{s^2 + 2\tilde{\zeta}_k \tilde{\omega}_k s + \tilde{\omega}_k^2} \tag{2}$$

where σ_0 is a positive real constant. The relations between the natural frequencies and damping ratios in Eq. (1), ω_k and ζ_k, to those in Eq. (2), $\tilde{\omega}_k$ and $\tilde{\zeta}_k$, are as follows. From Fig. 1 it is clear that

$$\tilde{\omega}_k^2 = \omega_k^2(1 - \zeta_k^2) + (\zeta_k \omega_k - \sigma_0)^2 \ge (\omega_k - \sigma_0)^2$$

$$\text{for} \quad \zeta_k < 1 \quad \text{and} \quad \omega_k > \sigma_0$$

and, hence, that $1/\tilde{\omega}_k \le 1/\omega_k - \sigma_0$. This bound can be used only for terms whose real part is located to the left of the shifted imaginary axis, i.e., $\zeta_k \omega_k > \sigma_0$.

Because $\sigma_0 > 0$, the range of the damping ratio $\tilde{\zeta}_k$ is changed from the range of ζ_k; it is clear from Fig. 1 that $\tilde{\epsilon}_1 \le \epsilon_1$ and $\tilde{\epsilon}_2 \le \epsilon_2$. Using the definition $\cos[\phi_k] = \zeta_k$, bounds are obtained for the modified range $[\tilde{\epsilon}_1, \tilde{\epsilon}_2]_k$ for the damping factor $\tilde{\zeta}_k$. For each fixed k, it is true that

$$\cos[\phi_k] \ge \min_{k>n}\left[(\zeta_k \omega_k - \sigma_0)/\tilde{\omega}_k\right] = (\epsilon_1 \omega_k - \sigma_0)/\tilde{\omega}_k = \tilde{\epsilon}_{1,k}$$

$$\cos[\phi_k] \le \max_{k>n}\left[(\zeta_k \omega_k - \sigma_0)/\tilde{\omega}_k\right] = (\epsilon_2 \omega_k - \sigma_0)/\tilde{\omega}_k = \tilde{\epsilon}_2$$

Note that both $\min_{k>n}[\tilde{\epsilon}_1]$ and $\min_{k>n}[\tilde{\epsilon}_2]$ occur when $k = n+1$ and that the sector in the second quadrant defined by the pair $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)_k$ is being reoriented toward the imaginary axis. The modified range is thus defined by $\tilde{\epsilon}_1 = (\epsilon_1 \omega_{n+1} - \sigma_0)/\omega_{n+1}$ and $\tilde{\epsilon}_2 = (\epsilon_2 \omega_{n+1} - \sigma_0)/\omega_{n+1}$. Note that $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)_k \to (\epsilon_1, \epsilon_2)$ as $k \to \infty$, for $|\sigma_0|$ finite.

Because we have developed a transfer function mapping between $\sigma_0 = 0$ and $\sigma_0 > 0$ cases, for ease of presentation we shall use the notation of Eq. (1) for the remainder of this Note. That is, the complex variable s is also a function of σ_0.

A typical approximate rational transfer function $G_n(s = j\omega)$ includes the first n terms in Eq. (1), and, thus, the error $E(j\omega)$ is

$$E(j\omega) = G(j\omega) - G_n(j\omega) = \sum_{k=n+1}^{\infty} \frac{\delta_k}{(\omega_k^2 - \omega^2) + j2\zeta_k \omega_k \omega} \tag{3}$$

Remark

Large models of flexible structures, say with 500 modes, can be transformed into a transfer function form. The modal coefficients in Eq. (1) would then be obtained directly from the state, control, and feedback matrices in the state space form.

Uniform Bound

The modulus of the kth term of the FRF [Eq. (3)] is given by

$$|E_k(j\omega)| = \frac{\delta_k}{\sqrt{(\omega_k^2 - \omega^2)^2 + (2\zeta_k \omega_k \omega)^2}} = \frac{\delta_k}{\omega_k^2 \sqrt{f_k}}$$

where

$$f_k(\omega) = [1 - \omega^2/\omega_k^2]^2 + [2\zeta_k\omega/\omega_k]^2$$

The modulus $|E_k(j\omega)|$ has a maximum in $\omega\in(-\infty,\infty)$ exactly when $f_k(\omega)$ has a minimum. A simple algebraic rearrangement of f_k yields[5]

$$f_k(\omega) = \left[\omega^2 - \omega_k^2(1-2\zeta_k^2)\right]^2/\omega_k + 4\zeta_k^2(1-\zeta_k^2), \quad \omega\in(-\infty,\infty)$$

and clearly the minimum of $f_k(\omega)$ occurs when $\omega = \omega_k\sqrt{1-2\zeta_k^2}$. Thus, the uniform bound for the modulus of a single FRF $E_k(j\omega)$ is

$$|E_k(j\omega)| \leq \frac{\delta_k}{2\omega_k^2\zeta_k\sqrt{1-\zeta_k^2}}, \qquad \omega\in(-\infty,\infty)$$

The uniform bound for the approximation error modulus is thus defined as

$$|E(j\omega)| \leq \frac{\delta}{\Gamma_1}\sum_{k=n+1}^{\infty}\frac{1}{\omega_k^2} = R_1, \qquad \omega\in(-\infty,\infty) \tag{4}$$

where

$$\delta = \max_{k>n}|\delta_k|$$

$$\Gamma_1 = \min_{k>n}\left[2\zeta_k\sqrt{1-\zeta_k^2}\right] > 0$$

The series converges only if $\rho > 0.5$ and the sequence $\{c_k\}$ is bounded above from zero.

It is common practice in control synthesis to assume that the modal damping factor ζ_k is constant for all terms (modes). However, this assumption is not supported by experiments, and so it becomes necessary to employ error bounds similar to the uniform bound, which allows a wide variation in the damping ratio for different modes. To compute the bound [Eq. (4)], it is sufficient to know the constants ϵ_1 and ϵ_2, giving

$$\Gamma_1 = \min_i\left[2\epsilon_i\sqrt{1-\epsilon_i^2}\right], \qquad i = 1,2$$

Frequency Dependent Bound

A bound that approches zero as the frequency approaches infinity can be derived for the FRF [Eq. (3)]. Consider the modulus of the kth term rearranged to

$$|E_k(j\omega)| = \frac{\delta_k}{\omega_k\,\omega\sqrt{f_k(\omega)}}$$

where

$$f_k(\omega) = (\omega_k^2 - \omega^2)^2/(\omega_k\omega)^2 + 4\zeta^k$$

The modulus $|E_k(j\omega)|$ can be bounded above using the inequality

$$|E_k(j\omega)| \leq \delta/\left(\omega\omega_k \min_\omega\left[\sqrt{f_k(\omega)}\right]\right)$$

for a fixed k. The minimum of $f_k(\omega)$ occurs when $\omega = \omega_k$ giving

$$|E_k(j\omega)| \leq \frac{\delta_k}{2\omega\omega_k\zeta_k}, \qquad \omega\in(-\infty,\infty)$$

The frequency dependent bound for the approximation error modulus is thus defined as

$$|E(j\omega)| \leq \frac{\delta}{\Gamma_2\omega}\sum_{k=n+1}^{\infty}\frac{1}{\omega_k} = R_2(\omega), \qquad \omega\in(-\infty,\infty) \tag{5}$$

where

$$\delta = \max_{k>n}|\delta_k| \qquad \text{and} \qquad \Gamma_2 \min_{k>n}[2\zeta_k] = 2\epsilon_1 > 0$$

Overall Bound

The smaller of the bounds R_1 and $R_2(\omega)$ can be used over different frequency ranges to produce a tighter overall bound (Fig. 2). Note that, as expected, as more terms (modes) are included in the approximate model, both bounds decrease with limit zero as $n\to\infty$. The size of the error bound should be used to indicate whether a certain open-loop approximate model contains a sufficient number of vibration modes.

Numerical Computation of the Bounds

The numerical computation of the bounds [Eqs. (4) and (5)] is most often straightforward. Sums for series with $\sigma_0 = 0$ are tabulated for different integer powers ρ in many mathematical handbooks.[6] For ρ positive real, the integral test[7] can be used

$$\int_n^{m+1} x^{-\rho}\,dx \leq \sum_n^m k^{-\rho} \leq n^{-\rho} + \int_n^m x^{-\rho}\,dx$$

for some integers k, n, and m. However, additional work is required in order to compute the bounds when $\sigma_0 > 0$. To do so, consider the series

$$\sum_{k=n+1}^{\infty}\frac{1}{k^\rho - \sigma_0} = \sum_{k=n+1}^{\infty}\frac{k^\rho}{k^\rho-\sigma_0}\frac{1}{k^\rho}$$

$$\leq \max_{k>n}\left[|k^\rho/(k^\rho-\sigma_0)|\sum_{k=n+1}^{\infty}\frac{1}{k^\rho}\right]$$

where $\rho > 1$, σ_0 is a positive constant, and $k^\rho > \sigma_0$. The inequality implies, by the comparison test,[7] that the series on the left is absolutely convergent. The sum for $\sigma_0 > 0$ is bounded above by the sum for $\sigma_0 = 0$ multiplied by the factor

$$\max_{k>n}\left[|k^\rho/(k^\rho-\sigma_0)|\right] \to 1 \qquad \text{as} \qquad k\to\infty$$

Thus, in a series truncated after n terms,

$$\max_{k>n}\left[|k^\rho/(k^\rho-\sigma_0)|\right] = (n+1)^\rho/\left[(n+1)^\rho - \sigma_0\right]$$

Graphical Interpretation of the Error Bounds

The polar (Nyquist) plot $G_n(j\omega)$ is drawn together with error circles associated with each frequency in that range. The polar plot of $G(j\omega)$ is then found within a tube of uncertainty, defined by the union of all the smaller error circles. The tube of uncertainty corresponds, in an abstract sense, to Gershgorin disks.[8] Such a Nyquist plot for a certain robust control scheme for a Bernoulli-Euler beam is shown in Fig. 3.[9] Clearly, the developed error bounds [Eqs. (4) and (5)] allow synthesis of robust compensator for the nonrational system using Bode and

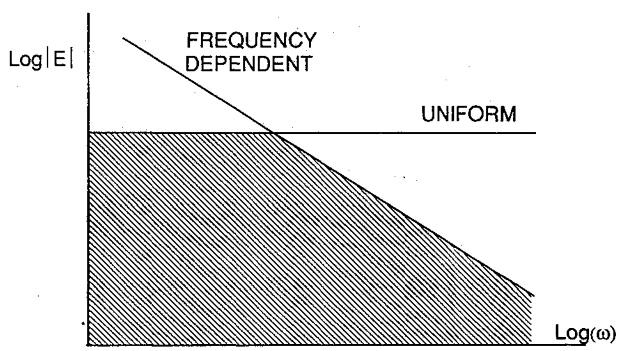

Fig. 2 Approximation error bounds.

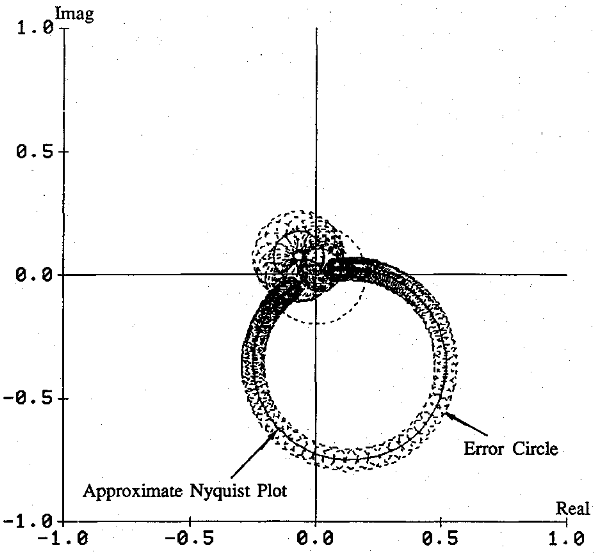

Fig. 3 Approximate Nyquist plot with error circles.[9]

Nyquist plots of the approximate system and the error bounds. For several illustrative examples see Chait et al.[2] and Chait.[9]

Numerical Example

Consider a transfer function of the form of Eq. (1) where $\omega_k = (k\pi)^2$, $\delta = 2$, $\epsilon_1 = 0.005$, and $\epsilon_2 = 0.5$. This transfer function corresponds to the ratio between a position point sensor to a point actuator for the Bernoulli-Euler beam with unity parameters.[2,9]

The uniform bound R_1 and the frequency dependent bound $R_2(\omega)$ can be calculated using Eqs. (4) and (5) for $\sigma_0 = 0$. For this system, we have $\rho = 2$, $\delta = 2$, and $\Gamma_1 = \Gamma_2 = 0.01$. For a truncated series that consists of the first term alone ($n = 1$):

$$R_1 = 200 \sum_{k=2}^{\infty} \frac{1}{(k\pi)^4} \approx 0.1692$$

$$R_2(\omega) = (200/\omega) \sum_{k=2}^{\infty} \frac{1}{(k\pi)^2} \approx 13.16/\omega$$

for $n = 2$, $R_1 = 0.041$ and $R_2(\omega) = 8.106/\omega$; for $n = 4$, $R_1 = 0.0075$ and $R_2(\omega) = 4.59/\omega$; and for $n = 10$, $R_1 = 0.00000373$ and $R_2(\omega) = 2.02/\omega$.

For the transfer function considered in this example with shifted imaginary axis by $\sigma_0 = 0.1$, similar bounds can be computed using Eqs. (4) and (5). For $n = 1$, since $\zeta_2\omega_2 > 0.1$, the range of $\tilde{\zeta}_k$ is found to be $\tilde{\epsilon}_1 = 0.00246$ and $\tilde{\epsilon}_2 = 0.497$. Thus, the bounds for $n = 1$ are $R_1 = 0.344$ and $R_2(\omega) = 26.79/\omega$. For $n = 10$, since $\zeta_{11}\omega_{11} > 0.1$, we compute $\tilde{\epsilon}_1 = 0.0049$ and $\tilde{\epsilon}_2 = 0.497$, and the modified bounds are $R_1 = 0.00000381$ and $R_2(\omega) = 2.06/\omega$. Note that the bounds become larger as the shifted axis is closer to the $(n+1)$th root. Consequently, the closer the desired closed-loop decay rate is to the decay rate of the open-loop modes, the more costly is the control solution.

Conclusions

Several useful formulas were developed for computing approximation error bounds for certain models of large space structures. These are the uniform and frequency dependent bounds about the imaginary axis and about a shifted imaginary axis. It was shown that these bounds can capture arbitrary variations of the damping factor within a specified range and allow uncertainties in the location of the actuators and sensors. In addition, these bounds can be computed along any vertical axis in the open left half-plane, which is necessary for control synthesis for a specified decay rate. They can be employed, in robust control techniques, to obtain safe estimates of closed-loop frequency responses corresponding to systems that have either infinite degrees of freedom or a high-order model. Numerical computation aspects and graphical interpretation are discussed.

References

[1]Chen, M. J., and Desoer, C. A., "Necessary and Sufficient Condition for Robust Stability of Linear Distributed Parameter Feedback Systems," *International Journal of Control,* Vol. 35, No. 2, 1982, pp. 255-
267.

[2]Chait, Y., Radcliffe, C. J., and MacCluer, C. R., "Frequency Domain Stability Criterion for Vibration Control of the Bernoulli-Euler Beam," *Journal of Dynamic Systems, Measurement, and Control,* Vol. 110, No. 3, 1988, pp. 303-307.

[3]Garg, S. C., "Frequency Domain Analysis of Flexible Spacecraft Dynamics," *Proceedings of the Second VPI&SU/AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft,* edited by L. Meirovitch, Virginia Polytechnic Inst. and State Univ., Blacksburg, VA, June 1979, pp. 561-575.

[4]Yedavalli, R. M., "Critical Parameter Selection in the Vibration Suppression of Large Space Structures," *Journal of Guidance, Control, and Dynamics,* Vol. 7, No. 3, 1984, pp. 274-278.

[5]Ogata, K., *Modern Control Engineering,* Prentice-Hall, Englewood Cliffs, NJ, 1970.

[6]Beyer, W. H., *CRC Standard Mathematical Tables,* CRC Press, Boca Raton, FL, 1982.

[7]Trench, W. F., *Advanced Calculus,* Harper & Row, New York, 1978.

[8]Franke, D., "Application of Extended Gershgorin Theorems to Certain Distributed-Parameter Control Problems," *Proceedings of the 24th Computers Decision and Control Conference,* 1985, pp. 1151-1156.

[9]Chait, Y., "Frequency Domain Robust Control of Distributed Parameter Systems," Ph.D. Dissertation, Michigan State University, East Lansing, MI, 1988.

Orbital Motion Under Continuous Radial Thrust

Frederick W. Boltz*
NASA Ames Research Center,
Moffett Field, California 94035

Introduction

In general, the problem of orbital motion of spacecraft under the disturbing influence of continuous thrust produced by rocket propulsion requires numerical methods for solution. However, for a vehicle initially in a nearly circular orbit, when the thrust acceleration is constant in magnitude and directed either radially or tangentially, some analytical results of practical interest may be obtained.[1-4] A comprehensive analysis of orbital motion under continuous low thrust is presented by Battin[1,2] from numerical solutions and from analytic solutions originally given by Tsien,[3] for the case of constant radial acceleration, and by Benney,[4] for the case of constant tangential acceleration.

The analytic solution obtained for the problem of constant radial acceleration being applied to a vehicle in circular orbit reveals an interesting result. It is found that there is a critical value of constant radial acceleration above which escape speed will eventually be attained and below which the vehicle will simply spiral out to a higher altitude and then return, with continuation of thrust, to the initial altitude. This analytic

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*Aerospace Engineer (retired). Member AIAA.